On the structure of polytopes related to the Hamilton Cycle Problem

by

Sogol Mohammadian

A thesis submitted in fulfilment

of the requirements for the degree of

Doctor of Philosophy

Supervisor: Thomas Kalinowski

Associate Supervisor: Eric Beh

THE UNIVERSITY OF
NEWCASTLE
AUSTRALIA

The University of Newcastle

School of Mathematical and Physical Sciences

December, 2019

To my mother

Statements

Statement of Originality

I hereby certify that the work embodied in the thesis is my own work, conducted under normal supervision. The thesis contains no material which has been accepted, or is being examined, for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made. I give consent to the final version of my thesis being made available worldwide when deposited in the University's Digital Repository, subject to the provisions of the Copyright Act 1968 and any approved embargo.

Statement of Authorship

I hereby certify that the work embodied in this thesis contains published papers of which I am a joint author. I have included as part of the thesis a written statement, endorsed by my supervisor, attesting to my contribution to the joint publications.

Sogol Mohammadian
December 2019

By signing below I confirm that the candidate contributed significantly to the publications entitled "Hamiltonian cycles and subsets of discounted occupational measures" [22] and "Feasible bases for a polytope related to the Hamilton Cycle Problem" [38]. In particular, she developed proofs for major results, drafted the manuscripts, and revised them critically.

Thomas Kalinowski
December 2019

Acknowledgments

Firstly, I would like to express my sincere gratitude to my supervisor Dr Thomas Kalinowski for his continuous support during my PhD journey, for his patience, motivation, and immense knowledge. I could not have imagined having a better advisor and mentor.

I am genuinely thankful to Prof. Catherine Greenhill for supporting me to visit her at UNSW. I am grateful to Dr Ali Eshragh and Prof. Jerzy Filar for their willingness to collaborate with me. Their knowledge and advice indeed helped me to grow.

I gratefully acknowledge the University of Newcastle and School of Mathematical and Physical Sciences for funding my PhD through international postgraduate research scholarship scheme and providing me with this opportunity to do my research.

I wish to thank my friends and fellows in Newcastle who fulfilled my time with joy and happiness. Special thanks to Mahdi Abolghasemi who has been by my side throughout this journey, and always encouraged me.

My deepest gratitude is for my mother and my family whose support, faith, and love has been the the inspiration of my life.

Sogol Mohammadian
The University of Newcastle
December 2019

Contents

List of Tables

List of Figures

Notation and Acronyms

Notation

A given digraph on nodes

Set of all nodes in

Set of all arcs in

Set of all nodes in a given graph

Set of all arcs in a given graph

Set of all in-neighbourhoods for node

Set of all out-neighbourhoods for node

Total number of in-neighbourhoods for node (in-degree for node)

Total number of out-neighbourhoods for node (out-degree for node)

Total number of arcs incident with node

Total inflow for node

Total outflow for node

Acronyms

GNF	Generalised Network Flow
HCP	Hamilton Cycle Problem
MDP	Markov Decision Process
TSP	Travelling Salesman Problem

Abstract

The *Hamilton Cycle Problem* (HCP) is one of the classical problems in combinatorics. The problem is to decide if a given graph G contains a cycle that visits each node exactly once. Cycles that pass through every node of a graph exactly once are called *Hamilton cycles*. If a graph contains at least one Hamilton cycle, then it is called *Hamiltonian*. Otherwise, it is *non-Hamiltonian*.

It is known that the HCP is NP-complete, so it is unlikely that there is an exact algorithm which terminates in polynomial time and solves the problem in general. However, there are many very successful heuristics, and a lot of research has been done towards a theoretical understanding of the performance of these algorithms. In particular, there is an extensive literature on algorithms for finding Hamilton cycles in random graphs. Nevertheless, some questions in this area are still open (see, for example, [28]).

In 1994, Filar and Krass [24] proposed a new approach to the HCP based on the theory of Markov Decision Processes (MDPs). Their approach initiated a new line of research studying various closely related polytopes constructed from an input graph G. In particular, two polytopes depending on a parameter β, $0 < \beta < 1$, were introduced in [21]. We refer to these polytopes as $\mathcal{H}_\beta(G)$ and $\mathcal{WH}_\beta(G)$. Their points correspond to discounted occupational measures induced by the MDP policies.

There is a correspondence between the Hamilton cycles of a given graph G and specific extreme points of the polytopes mentioned above. As a consequence, these polytopes can be utilised as the basis for a sampling-based random walk algorithm to search for Hamilton cycles. The essential conditions for the efficiency of this approach are (i) existence of sufficiently many extreme points corresponding to Hamilton cycles, (ii) rapid mixing of the random walk, and (iii) polynomial run-time for a single step in the random walk. The third condition is always satisfied for the walk on the feasible bases as in this case the steps are, in fact, pivots in the simplex algorithm.

In this thesis, we make the observation that if we assume two widely believed conjectures in complexity theory ($P \neq NP$ and $P = BPP$) then the three necessary conditions cannot hold in full generality. However, this does not rule out that the approach works for particular classes of graphs. Interestingly, the algorithm from [30] for the random regular graph can be interpreted as a sampling-based search algorithm on the nodes of the polytope $\mathcal{WH}_1(G)$ which is obtained by setting $\beta = 1$ in the definition of $\mathcal{WH}_\beta(G)$. This gives rise to the following question, which motivates the

research presented in this thesis: Does sampling extreme points (or feasible bases) of polytopes , and solve any of the algorithmic questions for random graphs that are still open? In this thesis, we make some progress towards answering this question by establishing combinatorial results on the structure of these polytopes.

Results on : We characterise the feasible bases of this polytope for a general input graph. Furthermore, we determine the expected numbers of different types of extreme points and feasible bases when the underlying graph is random. Our results indicate that the total number of feasible bases of the polytope grows exponentially faster than the number of Hamiltonian bases for an input random graph, and a similar result is true if we consider extreme points instead of feasible bases. As a consequence, sampling extreme points or feasible bases of this polytope cannot lead to an efficient sampling-based algorithm. We present some computational evidence that the behaviour of is more promising, which provides the motivation for a more in-depth study of this polytope.

Results on : We present two general results about the structure of feasible bases for the polytope . First, we show that the set of feasible bases is independent of when the parameter is close to one. This ensures that we can study the combinatorics of the polytope without worrying about numerical issues caused by values of very close to 1. Second, we show that can be interpreted as a generalised network flow polytope. This connects the polytope to some classical combinatorial optimisation, and allows us to prove strong statements about the combinatorial structure of the feasible bases. For a particular class of feasible bases, we provide a complete characterisation, and we deduce a recursion formula for the number of feasible bases of a particular type within this class.

Results on : We discuss that sampling extreme points of this polytope corresponds to sampling extreme points of the perfect matching polytope for an auxiliary graph associated with the input graph . Thus, the known results about sampling perfect matchings can be used for sampling extreme points of this polytope. Besides, we demonstrate that the number of Hamiltonian extreme points of this polytope is within a polynomial factor of the total number of extreme points when the input graph is a moderately dense random directed graph. Also, we show that the expected number of Hamiltonian bases is within a polynomial factor of the expected total number of feasible bases.

Publications

This thesis is written under the supervision of Dr Thomas Kalinowski and Prof. Eric Beh at the University of Newcastle. The research presented in this thesis led to one publication, and the submission of one additional journal article (which, at the time of writing, was under review).

P1 (see [22]) A. Eshragh, J. A. Filar, T. Kalinowski, and S. Mohammadian. Hamiltonian cycles and subsets of discounted occupational measures. Mathematics of Operations Research, 2019.

P2 (see [38]) T. Kalinowski and S. Mohammadian. Feasible bases for a polytope related to the Hamilton Cycle Problem. arXiv:1907.12691, 2019.

Chapter 3 is adapted from [22] and some parts of Chapter 4 were presented in [38].